15th Week Wednesday

31 Mayıs 2023 Çarşamba 11:28

June, 14^{th}

9:30 – 11:30

comp. ← TR

sofw. EE ← EN

9:30 — 2^{nd} Midt (forms); Final Exam ((Webwork) 50–60 min, 8–10 questions)

10:30 — Final Exam (; 2^{nd} Midt (Forms) → 40–45 min, 6–7 questions

10:40

2^{nd} Midt Topics : [Subspaces, Eigenvalue/vectors]

Final Exam : comprehensive

5^{th} Quiz → 16:00–16:40 → eigenvalue / eigenvector

6^{th} Quiz → Friday's lecture

Orthogonal (Dual) Complements of Subspaces

$S \leqslant \mathbb{R}^n$ a subspace $\qquad S^{\perp} = \{ \vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{s} = 0 \ \ \forall \vec{s} \in S \}$

! $S \cap S^{\perp} = \{ \vec{0} \}$

! $\dim(S) + \dim(S^{\perp}) = \dim(\mathbb{R}^n)$

Ex $\mathbb{R}^3 \to S = \operatorname{span}(e_1) = \{ (r,0,0) : r \in \mathbb{R} \} \leqslant \mathbb{R}^3$. Find a basis for $S^{\perp}$.

$\dim(\mathbb{R}^3) = 3$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ $\quad \boxed{\dim(S) = 1}$ ✓

Orthogonal complement of S → $S^{\perp} = \{ (x,y,z) : (x,y,z) \cdot (1,0,0) = 0 \} = \{ (0,r,s) : r,s \in \mathbb{R} \}$

→ $x \cdot 1 + y \cdot 0 + z \cdot 0 = 0$ → $y = r \in \mathbb{R}$, $z = s \in \mathbb{R}$, $x = 0$

a system of lin. eqns. 3 unknowns 1 eqn

$\begin{bmatrix} 0 \\ r \\ s \end{bmatrix} = r \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

A basis for $S^{\perp} = \left\{ \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{e_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{e_3} \right\}$ $\quad \boxed{\dim(S^{\perp}) = 2}$ ✓

! **Note**: In order to be orthogonal to all vectors in S, it is necessary and sufficient to be orthogonal to all basis vectors.

Ex $S = \operatorname{span} \left\{ \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} \right\} \leqslant \mathbb{R}^3$. Find a basis for $S^{\perp}$.

Let S be the subspace generated by $\left\{ \begin{bmatrix} 3 \\ 5 \\ ? \end{bmatrix} \right\}$ → $\dim(S) = 1$

Let S be the subspace generated by $\left\{ \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} \right\}$ $\rightarrow$ $\dim(S) = 1$

$$S^\perp = \{ (x,y,z) : \underline{(x,y,z) \cdot (3,5,-7) = 0} \} \leqslant \mathbb{R}^3$$

$\rightarrow$ $3x + 5y - 7z = 0$ $\quad$ $\left.\begin{matrix} y = r \in \mathbb{R} \\ z = s \in \mathbb{R} \end{matrix}\right\}$ free $\quad$ $x = \dfrac{7s - 5r}{3}$

$$S^\perp = \left\{ \underbrace{\left(\frac{7s-5r}{3}, r, s \right)}_{} : r, s \in \mathbb{R} \right\}$$

$$\begin{bmatrix} \frac{7s-5r}{3} \\ r \\ s \end{bmatrix} = r \begin{bmatrix} -5/3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 7/3 \\ 0 \\ 1 \end{bmatrix}$$

A basis for $S^\perp = \left\{ \begin{bmatrix} -5/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\dim(S^\perp) = 2$

Ex/ Let S be the subspace of $\mathbb{R}^4$ generated by the vectors $(1, 0, -2, 1)$ and $(0, 1, 3, -2)$. Find a basis for $S^\perp$.

$S = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix} \right\}$ $\quad$ $\dim(S) = 2$.

$$S^\perp = \left\{ \underline{(x,y,z,t)} : \begin{matrix} (x,y,z,t) \cdot (1,0,-2,1) = 0 \text{ and} \\ (x,y,z,t) \cdot (0,1,3,-2) = 0 \end{matrix} \right\}$$

$$\left.\begin{matrix} 1.x + 0.y + -2z + 1.t = 0 \\ x.0 + y.1 + 3.z - 2t = 0 \end{matrix}\right\} \rightarrow \begin{array}{c} \begin{matrix} x & y & z & t & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{array}\right] \end{array}$$

a system of lin eqns.
2eqns + 4 unknowns

$z = r \in \mathbb{R}$
$t = s \in \mathbb{R}$
$\Rightarrow y = 2s - 3r$
$\Rightarrow x = 2r - s$

$$S^\perp = \{ (2r-s, 2s-3r, r, s) : r, s \in \mathbb{R} \}$$

$$\begin{bmatrix} 2r-s \\ 2s-3r \\ r \\ s \end{bmatrix} = r \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

A basis for $S^\perp = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\dim(S^\perp) = 2$

Ex $S = \{ (a+b,\ 2a-b,\ 3a) : a, b \in \mathbb{R} \} \leqslant \mathbb{R}^3$

Find a basis for $S^\perp$.

we should first find a basis for S :

$$\begin{bmatrix} a+b \\ 2a-b \\ 3a \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

A basis for $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

$$S^\perp = \left\{ (x,y,z) : \begin{array}{l} (x,y,z)\cdot(1,2,3) = 0 \text{ and} \\ (x,y,z)\cdot(1,-1,0) = 0 \end{array} \right\}$$

$$\begin{array}{l} x+2y+3z=0 \\ x-y+0z=0 \end{array} \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right]$$

$$\to \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array} \to z = r \in \mathbb{R} \Rightarrow y = -r$$

$x + 2r + 3r = 0 \Rightarrow x = -r$

$S^\perp = \{ (-r, -r, r) : r \in \mathbb{R} \}$

A basis for $S^\perp = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Orthogonal Basis :

Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$ is called "orthogonal" $\Leftrightarrow$

$$\forall i,j \in \{1,\dots,n\},\quad \vec{v}_i \cdot \vec{v}_j = 0$$

Ex $\left\{ \underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}}_{v_3} \right\}$ is a basis for $\mathbb{R}^3$, Is this an orthogonal basis?

$\vec{v}_1 \cdot \vec{v}_2 = 2.1 + -1.2 + 0 = 0$ ✓

$\vec{v}_1 \cdot \vec{v}_3 = 2.0 + -1.0 + 0.3 = 0$ ✓

$\vec{v}_2 \cdot \vec{v}_3 = 1.0 + 2.0 + 0.3 = 0$ ✓

Yes.

$\|v_1\| = \sqrt{2^2 + -1^2 + 0} = \sqrt{5} \neq 1$

not orthonormal

Orthonormal Basis :

Let $\{v_1, v_2, \dots, v_n\}$ be an orthogonal basis for $\mathbb{R}^n$, it is called "orthonormal basis" $\Leftrightarrow$ $\boxed{\|v_i\| = 1,\ \forall i}$

$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$
$$\left\{ \begin{bmatrix} 1 \\ \end{bmatrix} \begin{bmatrix} 0 \\ \end{bmatrix} \begin{bmatrix} 0 \\ \end{bmatrix} \right.$$

Ex $S = \text{span}\left\{ \begin{array}{cccc} v_1 & v_2 & v_3 \\ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, & \begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \end{bmatrix}, & \begin{bmatrix} 0 \\ 5 \\ -1 \\ 0 \end{bmatrix} \end{array} \right\}$

↓ a basis for S. an orthogonal basis? an orthonormal basis? ✗

NO!

$v_1 \cdot v_2 = 1.0 + 0.3 + 0.0 + (-1)(-2) = 2 \neq 0$

$v_2 \cdot v_3$

$v_1 \cdot v_3$

Orthonormalization:

Ex $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$.

but not orthonormal.

$v_1 \quad v_2 \quad v_3$

$\|v_1\| = \sqrt{2^2+(-1)^2+0} = \sqrt{5}$

$\|v_2\| = \sqrt{1^2+2^2+0} = \sqrt{5}$

$\|v_3\| = \sqrt{0+0+9} = 3$

How do we make it an orthonormal one?

$\left\{ \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 0\right), \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right), \left(0, 0, \frac{3}{3}\right) \right\} \rightarrow$ ✓ orthonormal basis.

If $\{v_1, v_2, \dots, v_n\}$ is an orthogonal basis but not an orthonormal basis then,

we scalar multiply each $\vec{v}_i$ with $\frac{1}{\|v_i\|}$.

$\Rightarrow \left\{ \frac{\vec{v}_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \dots, \frac{\vec{v}_n}{\|v_n\|} \right\} \rightarrow$ this is going to be an orthonormal basis.

Orthogonalization (Gram-Schmidt Orthogonalization Process)

Given an arbitrary basis $\{x_1, x_2, \dots, x_n\}$ which is not orthogonal

$\rightarrow \vec{y}_1 = \vec{x}_1$

$\{y_1, y_2, \dots, y_n\}$

↓ orthogonal basis

$\rightarrow \vec{y}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot y_1}{y_1 \cdot y_1} \vec{y}_1$

$\vec{y}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1} \cdot \vec{y}_1 - \frac{\vec{x}_3 \cdot \vec{y}_2}{\vec{y}_2 \cdot \vec{y}_2} \vec{y}_2$

$\vec{y}_4 = \vec{x}_4 - \frac{x_4 \; y_1}{y_1} y_1 \quad \frac{x_4 \; y_2}{} y_2 \quad \frac{x_4 \; y_3}{} y_3$

$$\vec{y}_n = \vec{x}_n - \sum_{i=1}^{n-1} \frac{\vec{x}_n \cdot \vec{y}_i}{\vec{y}_i \cdot \vec{y}_i} \cdot \vec{y}_i$$ ✓

Ex

$$S = \text{span}\left\{ \overset{\vec{x}_1}{\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}}, \overset{\vec{x}_2}{\begin{bmatrix}0\\3\\0\\-2\end{bmatrix}}, \overset{\vec{x}_3}{\begin{bmatrix}0\\5\\-1\\0\end{bmatrix}} \right\}$$

a basis for S. an orthogonal basis? an orthonormal basis? ✗

NO!

Create an orthogonal basis using Gram-Schmidt process.

$\vec{x}_2 \cdot \vec{y}_1 = 0.1 + 3.0 + 0.0 + (-2)(-1) = 2$

$\vec{y}_1 \cdot \vec{y}_1 = 1.1 + 0.0 + 0.0 + (-1)(-1) = 2$

$\vec{y}_1 = \vec{x}_1 = (1, 0, 0, -1)$ ✓

$\vec{y}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1} \vec{y}_1 = (0,3,0,-2) - (1,0,0,-1) = (-1, 3, 0, -1)$ ✓

$\vec{x}_3 \cdot \vec{y}_1 = 0.1 + 5.0 + -1.0 + 0.-1 = 0$

$\vec{x}_3 \cdot \vec{y}_2 = \cancel{0.-1} + 5.3 + \cancel{-1.0} + \cancel{0.-1} = 15$

$\vec{y}_2 \cdot \vec{y}_2 = -1.-1 + 3.3 + 0.0 + -1.-1 = 11$

$\vec{y}_3 = \vec{x}_3 - \cancel{\frac{\vec{x}_3 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1}\vec{y}_1} - \frac{\vec{x}_3 \cdot \vec{y}_2}{\vec{y}_2 \cdot \vec{y}_2}\vec{y}_2$

$= (0,5,-1,0) - \frac{15}{11}(-1,3,0,-1)$

$5 - \frac{45}{11}$

$= \left(\frac{15}{11}, \frac{10}{11}, -1, \frac{15}{11}\right)$ ✓

$\{y_1, y_2, y_3\} \rightarrow$ orthogonal.